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ON KERNELS IN *i*-TRIANGULATED GRAPHS

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A directed graph is said to be kernel-perfect if every induced subgraph possesses a kernel (independent, absorbing subset). A necessary condition for a graph to be kernel-perfect is that every complete subgraph C has an absorbing vertex (i.e., a successor of all vertices of C). In this work, we show that this condition is sufficient for i-triangulated graphs, where every odd cycle has two non-crossing chords.

This result appears as a special case of a general relationship between the notion of kernel-perfectness and the well known strong perfect graph conjecture of Berge.

1. The result

We consider finite, loopless graphs, without multiple edges. Undefined terms are in Berge [1].

A *chord* of an elementary cycle $[x_0, x_1, \ldots, x_{p-1}, x_0]$ is an edge $[x_i, x_j]$ $(i, j \leqslant p-1)$ such that the vertices x_i and x_j are not consecutive on the cycle. Two chords $[x_i, x_j]$ and $[x_k, x_m]$ of an elementary cycle, where $0 \leqslant i < j \leqslant p-1$ and $0 \leqslant k < m \leqslant p-1$ are assumed, are *crossing chords* if $i<k<j<m$ or $k<i<m<j$.

A graph is said to be *i-triangulated* if every elementary cycle of odd length (at least five) has two non-crossing chords. This notion, introduced by Gallai [9] is a common generalization of triangulated graphs (where every elementary cycle has a chord) and bipartite graphs. It is a part of a larger class of perfect graphs defined by Meyniel [12], where every elementary odd cycle has two chords. It is also natural to consider the so-called parity graphs (where every elementary odd cycle has two crossing chords) defined by Olaru and Sachs in [14].

If $D = (X, U)$ is a digraph (a finite, loopless, directed graph, without multiple arcs), we denote by D^* the underlying undirected simple graph. The digraph D, or the arc-set U is called an *orientation* of D^*. A *reversible* arc of U is an arc (x, y) such that (y, x) also belongs to U.

A *kernel* of a digraph $D = (X, U)$ is a subset $K \subset X$ such that:
K is independent : $K \cap \Gamma_D^-(K) = \emptyset$;
K is absorbing : $K \cup \Gamma_D^-(K) = X$.

A digraph D is *kernel-perfect* if every induced subgraph of D possesses a kernel. Thus every complete subgraph C of a kernel-perfect digraph must have an *absorbing vertex* (i.e., a successor of all other vertices of C), and by induction must have a *dominating vertex*, i.e., a predecessor of all other vertices of C.

A digraph D is called a *normal orientation* of D^* if every complete subgraph of D possesses an absorbing vertex (or equivalently a dominating vertex).

Remark. A kernel-perfect digraph is a normal orientation of its underlying undirected graph.

Conjecture A (C. Berge, P. Duchet (1982) [2]). *A graph G is perfect if and only if any normal orientation of G is kernel-perfect.*

Conjecture B (H. Meyniel [15]). *An orientation of a Meyniel graph is kernel-perfect when it satisfies the condition: every circuit of length* 3 *has two reversible arcs.*

The well-known Berge's strong perfect graph conjecture would imply the 'if' part of Conjecture A. One may also remark that Conjecture B is a particular case of Conjecture A, since an orientation that fulfills the condition of Conjecture B is necessarily a normal orientation.

Conjecture B has first been proved for parity graphs by Blidia [3, 4] and Duchet [7] in two different ways, and for i-triangulated graphs by Jacob [10]. It is now completely proved in a forthcoming paper (Blidia, Duchet, Maffray [5]).

For what concerns i-triangulated graphs, the result is now improved by the following theorem.

Theorem. *Let G be an i-triangulated graph. Then an orientation of G is kernel perfect if and only if it is a normal orientation.*

2. The proof

The *necessity* of the condition results from the remark above. For the *sufficiency* we use three previous results, by Gallai, Jacob and Richardson. We recall that the join G of n graphs $G_1, \ldots, G_n$ is obtained as follows: the vertex set of G is the disjoint union of vertex-sets of the G_i's; two vertices of G are adjacent in G if they are in different G_i's or if they are adjacent in one G_i.

Theorem (T. Gallai [9]). *If G is a connected i-triangulated graph, one of the following three properties is fulfilled*:

(i) *G has a cutset A which induces a complete subgraph of G*;

(ii) *G is the join of a complete graph and a bipartite graph*;

(iii) *G is the join of several independent sets* (*i.e., is a multipartite complete graph*).

Theorem (H. Jacob [10]). *Let a digraph D have a cutset A which induces a complete subgraph of D, and such that the graph $D-A$ has p connected components $B_1, B_2, \ldots, B_p$. Then D is kernel-perfect if and only if all the pieces of A (i.e., the subgraphs induced by $A \cup B_1$, $A \cup B_2, \ldots, A \cup B_p$) are kernel-perfect.*

Theorem (M. Richardson [13], short proof in [8]). *A digraph without odd circuits is kernel-perfect.*

A directed path of a digraph $D=(X, U)$ is considered as a sequence $(x_0, x_1, \ldots, x_q)$ of distinct vertices such that $(x_i, x_{i+1}) \in U$ $(i=0, 1, \ldots, q-1)$. It becomes a circuit if $x_0 = x_q$. When $P(x, y)$ and $Q(y, z)$ are disjoint paths, $P(x, y) + Q(y, z)$ denotes the path from x to z, respectively passing through $P(x, y)$ and $Q(y, z)$.

We now prove that an i-triangulated graph G with a normal orientation D is kernel-perfect. We proceed by induction on the number n of vertices of G. The result is obvious for $n=1$ or 2. We suppose the property is true for graphs having at most $n-1$ vertices. If G is not connected, we get immediately a kernel by putting together the vertices of the kernels of each connected component of D. If G is connected, we apply Gallai's theorem and break the proof into three cases:

Case 1. G has a cutset A which induces a complete subgraph of G. By induction, the pieces of A in D are kernel-perfect, and then by Jacob's theorem, D is kernel-perfect.

Case 2. G is the join of a bipartite graph B and a complete graph C. We make a second induction on the number of reversible arcs of B in D.

Step 1. B has no reversible arc: since B has no odd cycle, it has no odd circuit, then by Richardson's theorem it possesses a kernel N. If N is not a kernel of D, there exists a vertex x of C such that $\Gamma^+(x) \cap N = \emptyset$. Thus $N \subset \Gamma^-(x)$.

Let $P(x, w)$ be a maximal directed path of C, of origin x, that uses no reversible arc of C. We show that its extremity w gives a kernel of D. We know that w is a neighbour of any other vertex of D, because C is complete and joined to any vertex of B.

If $\{w\}$ is not a kernel, there exists a vertex u such that $(w, u) \in U$ and $(u, w) \notin U$. We show that this is not possible.

- if $u \in C-P$, this contradicts the maximality of P.
- if $u \in P$, let $P'(u, w)$ be the part of P from u to w. Then $P'(u, w) + (w, u)$ is an antisymmetrical circuit in C, which contradicts the normal orientation condition.
- if $u \in N$, then $(u, x) + P(x, w) + (w, u)$ is an anti-symmetrical circuit in the complete subgraph induced by $C \cup \{u\}$, which contradicts the normal orientation condition.
- if $u \in B-N$, then by the definition of N, u has a successor v in B. By the hypothesis, the arc (u, v) is not reversible, and $(u, v) + (v, x) + P(x, w) + (w, u)$

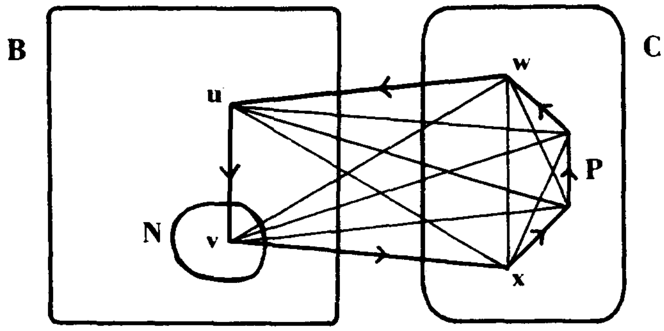

Fig. 1

is an antisymmetrical circuit in the complete subgraph induced by $C \cup \{u, v\}$, which contradicts the normal orientation condition (see Fig. 1). So we conclude that $\{w\}$ is a kernel of D.

Step 2. B possesses reversible arcs: Let (a, b) and (b, a) be a pair of reversible arcs of B. In the complete subgraph induced by $C_0 = C \cup \{a, b\}$, let x_1 be an absorbing vertex of C_0, then x_2 be an absorbing vertex of $C_0 - \{x_1\}$, then x_j be an absorbing vertex of $C_0 - \{x_1, x_2, \ldots, x_{j-1}\}$, up to the last vertex of C_0. We have $a = x_i$ and $b = x_j$ for some $i, j \leqslant |C_0|$, $i \neq j$. If, for example, $i < j$, we can delete $(x_i, x_j) = (a, b)$ from D. Deleting the arc (a, b) we obtain a graph D' which is the join of the bipartite graph $B' = B - (a, b)$ and the complete graph C. We show that D' still has a normal orientation. Indeed, for a complete subgraph C' of D', there are two possibilities:

– C' does not contain both a and b. Then C' has not been changed by the deletion of (a, b); and possesses an absorbing vertex like in D.

– C' contains a and b. Since B' has no triangle, C' contains no other vertex of B', and $C' \subset C_0 - (a, b)$. Thus x_k, where $k = \min\{l / x_l \in C'\}$ is an absorbing vertex of C' and D' has a normal orientation.

Since B' has less reversible arcs than B, the digraph D' has a kernel N by the induction hypothesis.

For conclusion, we need only remark that adding the arc (a, b) does not disturb absorption nor independence of N (for (b, a) is already in D'). Then N is a kernel of D.

Case 3. G is the join of several independent sets, say $S_1, S_2, \ldots, S_p$.

Let $x_i \in S_i$ $(i = 1, 2, \ldots, p)$. Then $C = \{x_1, x_2, \ldots, x_p\}$ is a complete subgraph of D. We can suppose for example that x_1 is a dominating vertex of C. By induction hypothesis, $D' = D - \{x_1\}$ has a kernel N. The graph D' is the join of $S_1 - \{x_1\}, S_2, \ldots, S_p$. A maximal independent subset of D' (and in particular a kernel) must be one of the sets S_i $(i \geqslant 2)$ or $S_1 - \{x_1\}$ (by definition of the join).

Then if $N = S_i$ $(i \geqslant 2)$, x_1 has x_i as successor in N, and N is a kernel of D. And if $N = S_1 - \{x_1\}$, then $N \cup \{x_1\} = S_1$ is a kernel of D.

This achieves the proof. $\square$

Concluding remark

Gallai's theorem gives a necessary condition for a graph to be i-triangulated. But this condition is not sufficient; consider for instance the odd cycles with one short chord.: they may be obtained by the way of Gallai's operations but they are not i-triangulated (some more sophisticated examples can be found).

Our result is in fact true for the class of graphs that can be constructed by iterated glueings along complete subgraphs of basic graphs: (1) the multipartite complete graphs, (2) the joins of a complete graph with a bipartite graph.

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References

[1] C. Berge, Graphs and Hypergraphs (North Holland, Amsterdam, 1976).
[2] C. Berge and P. Duchet, A perfect graph conjecture, Discrete Math., Problem Section, to appear.
[3] M. Blidia, Parity graphs with an orientation condition of Meyniel have a kernel, submitted.
[4] M. Blidia, Thèse de 3ème cycle, Paris, 1984.
[5] M. Blidia, P. Duchet and F. Maffray, On orientation of perfect graphs, submitted.
[6] M. Burlet and J. Fonlupt, Polynomial algorithm to recognize a Meyniel graph, in: (C. Berge and V. Chvâtal, eds.), Topics on Perfect Graphs (North Holland, Amsterdam, 1984).
[7] P. Duchet, On parity graph, submitted.
[8] H. Galeana Sánchez and V. Neumann-Lara, On kernels and semi kernels of digraphs, Discrete Math. 48 (1984) 67–76.
[9] T. Gallai, Graphen mit triangulierbaren ungeraden Vielecken, Magyar Tud. Akad. Mat. Kutato Int. Közl. 7 (1962) 3–36.
[10] H. Jacob, Thèse de 3ème cycle, Paris, 1979.
[11] F. Maffray, Sur l'existence de noyaux dans les graphes parfaits orientés, Thèse de 3ème cycle, Paris, 1984.
[12] H. Meyniel, On the perfect graph conjecture, Discrete Math. 16 (1976) 339–342.
[13] M. Richardson, Solutions of reflexive relations, Ann. Math. 58 (1953) 573–580.
[14] H. Sachs, On the Berge conjecture concerning perfect graphs, Combinatorial Structures and their Applications (Gordon and Breach, New York, 1970) 377–384.
[15] Séminaire du lundi, C. Berge, P. Rosenstiehl, Paris, 1982.